

# S-duality invariant perturbations in string cosmology

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## Abstract

We investigate the generation of curvature and isocurvature (dilaton, moduli and axion) perturbations in a general class of axion–dilaton–moduli models, including the pre–big bang scenario. Allowing for an arbitrary coupling constant  $\lambda$  between the dilaton field and the axion field, we exploit the  $SL(2, \mathbb{R})$  symmetry of the theory to obtain the spectral indices of the field perturbations in a pre–big bang type scenario. Axion field fluctuations about a homogeneous background field can yield a scale-invariant (Harrison-Zel’dovich) spectrum. As an example we present a string-motivated case with  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  symmetry, where a second axion field arises from the compactification of the ten-dimensional theory to four dimensions.

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# 1 Introduction

The cosmology of string theory is presently a very active area of research. The early universe provides one of the few areas where we may be able to place observational constraints on string theory. In particular one would like to know whether an inflationary expansion could occur in string theory which is capable of producing the initial conditions required by the standard hot big bang model. Recently, an alternative inflationary model, known as the *pre-big bang scenario*, has been developed based on the low-energy string effective action [1].

The fundamental assumption in the pre-big bang scenario is that the universe is initially in a weakly-coupled, low curvature regime. The non-minimal coupling between the dilaton and graviton can lead to a super-inflationary expansion, where the Hubble radius decreases with time and the universe evolves into a state of high curvature and strong coupling. This scenario has a number of interesting features compared with conventional inflation models. Perturbations arising from quantum fluctuations can be calculated exactly in many models without resorting to ‘slow-roll’ approximations and their amplitude may be normalised in the low curvature, weakly-coupled regime, where the vacuum state is well understood [2, 3, 4]. Moreover, there is no need to introduce a potential energy term during inflation <sup>1</sup>.

However, there are a number of pressing problems in this scenario which need to be addressed. Perhaps the most fundamental is the mechanism by which the singularity present in the high curvature regime can be avoided, bringing the superluminal expansion to an end (the Graceful Exit Problem) [6]. For recent work on possible resolutions see [7, 8]. Another important issue is whether density perturbations can be generated that are capable of seeding the observed large-scale structure in the universe. It is known that the dilaton and moduli fields produce perturbations with spectral index  $n = 4$ , making them too steep to be viable seeds for large-scale structure [2]. One possibility is that the electromagnetic perturbation spectrum generated in the pre-big bang could have interesting observational consequences [3].

Recently, Copeland, Easther and Wands [4] have shown that the coupling of the dilaton to the Neveu-Schwarz/Neveu-Schwarz (NS-NS) axion field in four dimensions can produce an axion spectrum which includes the scale-invariant  $n = 1$  case. The precise value depended on the expansion rate of the internal dimensions and varied between  $0.5 < n \leq 4$ . In that work, the role of the  $SL(2, R)$  ‘S-duality’ symmetry of the action was crucial. It was possible to construct the unique S-duality invariant field perturbations for the axion and dilaton fields and from them derive S-duality invariant solutions to the perturbation equations, valid when the axion field is time-dependent as well as in the case when it was frozen. In this paper we extend these ideas, considering the generation of perturbations about homogeneous background solutions in a more general class of models that may be relevant to the pre-big bang scenario. Once again we will find S-duality plays a significant role in enabling us to

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<sup>1</sup>For a more sceptical view of the pre-big bang scenario see [5]

obtain perturbation spectra for the fields.

We consider an effective Jordan–Brans–Dicke–type theory of gravity [9]:

$$S = \int d^4x \sqrt{-g} e^{-\Phi} \left[ R - \omega (\nabla \Phi)^2 - \frac{1}{2} e^{\Gamma \Phi} (\nabla \sigma)^2 - \frac{1}{2} (\nabla \beta)^2 \right], \quad (1.1)$$

where  $R$  is the Ricci scalar of the space–time with metric  $g_{\mu\nu}$  and  $g = \det g_{\mu\nu}$ . Our metric has signature  $(-, +, +, +)$ .  $\Phi$  represents the dilaton field,  $\beta$  is a ‘modulus’ field and  $\sigma$  is an ‘axion’ field, distinguished from the moduli by its explicit coupling to the dilaton. The axion-dilaton coupling is determined by the constant  $\Gamma \neq 0$ . The Brans-Dicke constant  $\omega$  determines the dilaton–graviton coupling. The region of parameter space where super–inflationary expansion in a pre–big bang phase may occur is  $-4/3 < \omega < 0$  [7].

An effective action such as that given in Eq. (1.1) can arise in various theories. One may consider the toroidal compactification of higher–dimensional Einstein gravity, i.e., Kaluza–Klein theory. If the gauge fields remain frozen and the internal space is isotropic with radius  $b$  and dimensionality  $d$ , the resulting four–dimensional action is given by the dilaton–graviton sector of Eq. (1.1) with  $\Phi \propto -d \ln b$  and  $\omega = -1 + 1/d$ . In the string effective action the dilaton already appears in the higher-dimensional theory and compactification to four-dimensions leads to a dilaton–graviton sector with  $\omega = -1$  plus extra moduli fields represented by the field  $\beta$  [10]. When  $\Gamma = 1$ ,  $\sigma$  may be viewed as a massless scalar field in the matter sector of the theory [11]. When  $\Gamma = 2$  the gradient of the pseudo-scalar axion field is dual to the field strength of the two–form potential arising in the NS–NS sector of the string effective action [12, 13]. Other couplings for an axion-type field are possible when moduli arising from the compactification of form fields on the internal space are excited [14].

This paper is organised as follows. In Section 2, we discuss the global  $\text{SL}(2, \mathbb{R})$  symmetry of action Eq. (1.1) and employ this symmetry to derive the general spatially flat Friedmann–Robertson–Walker (FRW) cosmology in Section 3. In Section 4, we derive the perturbation spectra of the three scalar fields that may arise from quantum fluctuations about this homogeneous background. In Section 5 we provide a string motivated example of the spectra obtained including moduli fields arising from compactification of the low–energy string action from ten to four dimensions on a six-dimensional torus. This possesses an additional  $\text{SL}(2, \mathbb{R})$  symmetry [15, 16]. We conclude in Section 6.

## 2 Global Symmetry of the Effective Action

The global symmetry of theory Eq. (1.1) becomes apparent in the ‘Einstein’ frame where the dilaton field is minimally coupled to gravity. This is obtained by the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega^2 \equiv e^{-\Phi}. \quad (2.1)$$

We will refer to the original frame with metric  $g_{\mu\nu}$  as the Jordan frame.<sup>2</sup> If we also redefine the dilaton field:

$$\phi \equiv (3 + 2\omega)^{1/2} \Phi , \quad (2.2)$$

the action Eq. (1.1) transforms to

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - \frac{1}{2} (\tilde{\nabla} \phi)^2 - \frac{1}{2} e^{2\lambda\phi} (\tilde{\nabla} \sigma)^2 - \frac{1}{2} (\tilde{\nabla} \beta)^2 \right] , \quad (2.3)$$

where the axion–dilaton coupling in the Einstein frame is

$$\lambda \equiv \frac{\Gamma}{2(3 + 2\omega)^{1/2}} . \quad (2.4)$$

Defining the symmetric  $2 \times 2$  matrix

$$N \equiv \begin{pmatrix} e^{\lambda\phi} & \lambda\sigma e^{\lambda\phi} \\ \lambda\sigma e^{\lambda\phi} & e^{-\lambda\phi} + \lambda^2 \sigma^2 e^{\lambda\phi} \end{pmatrix} \quad (2.5)$$

then implies that action Eq. (2.3) may be written as

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} + \frac{1}{4\lambda^2} \text{Tr} (\tilde{\nabla} N \tilde{\nabla} N^{-1}) - \frac{1}{2} (\tilde{\nabla} \beta)^2 \right] . \quad (2.6)$$

The matrix  $N$  is a member of the group  $\text{SL}(2, \mathbb{R})$ , and thus  $N^{-1} = -JN^T J$ , where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.7)$$

is the  $\text{SL}(2, \mathbb{R})$  metric. The dilaton and axion fields parameterise an  $\text{SL}(2, \mathbb{R})/\text{U}(1)$  coset and the action is invariant under a global  $\text{SL}(2, \mathbb{R})$  symmetry:

$$\tilde{g}_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} , \quad N \rightarrow \Theta N \Theta^T , \quad \beta \rightarrow \beta , \quad (2.8)$$

where

$$\Theta = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \quad (2.9)$$

is a member of  $\text{SL}(2, \mathbb{R})$  and hence  $ad - bc = 1$ .

The  $\text{SL}(2, \mathbb{R})$  group acts non-linearly on the complex scalar field  $\chi = \lambda\sigma + ie^{-\lambda\phi}$  such that the transformed field  $\chi \rightarrow (a\chi + b)/(c\chi + d)$ . Hence the transformed scalar fields are given by

$$e^{\lambda\phi} \rightarrow c^2 e^{-\lambda\phi} + (d + c\lambda\sigma)^2 e^{\lambda\phi} \quad (2.10)$$

$$\lambda\sigma e^{\lambda\phi} \rightarrow ace^{-\lambda\phi} + e^{\lambda\phi}(b + a\lambda\sigma)(d + c\lambda\sigma) . \quad (2.11)$$

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<sup>2</sup>For the examples which refer directly to string theory this will correspond to the string frame.

When  $\lambda\sigma = -d/c$  and  $c^2 = 1$  this reduces to  $\phi \rightarrow -\phi$  and is a strong-weak coupling “S-duality”. The modulus field remains invariant under the  $\text{SL}(2, \mathbb{R})$  transformation. The Einstein frame Lagrange density including the modulus field is also invariant under changes in the sign of the modulus field,  $\beta \rightarrow -\beta$ , or a global shift  $\beta \rightarrow \beta + \text{constant}$ .

Since Eqs. (2.1) and (2.2) correspond simply to field redefinitions, the  $\text{SL}(2, \mathbb{R})$  symmetry of the theory is also maintained in the Jordan frame. However the metric in the Jordan frame does change due to the change in the conformal factor:

$$\Omega^2 \equiv e^{-\Phi} \equiv e^{-\phi/\sqrt{3+2\omega}} \rightarrow \left[ c^2 e^{-\lambda\phi} + (d + c\lambda\sigma)^2 e^{\lambda\phi} \right]^{-1/(\lambda\sqrt{3+2\omega})} . \quad (2.12)$$

### 3 Homogeneous solutions

In this Section we present the general solutions for a spatially flat FRW cosmology [17], generalising earlier results found for specific couplings between the fields [13, 18]. These homogeneous axion-dilaton solutions can readily be extended to non-flat FRW models [13] and anisotropic models [11, 19, 20].

We will work in the Einstein frame where the line element is given by

$$d\tilde{s}^2 = \tilde{a}^2(\eta) \left( -d\eta^2 + dx^2 + dy^2 + dz^2 \right) , \quad (3.1)$$

where  $\eta$  is the conformally invariant time coordinate. The equations of motion are

$$\phi'' + 2h\phi' = \lambda e^{2\lambda\phi} \sigma'^2 , \quad (3.2)$$

$$\sigma'' + 2h\sigma' = -2\lambda\phi'\sigma' , \quad (3.3)$$

$$\beta'' + 2h\beta' = 0 , \quad (3.4)$$

$$h' + 2h^2 = 0 , \quad (3.5)$$

where  $h \equiv \tilde{a}'/\tilde{a}$  and a prime denotes differentiation with respect to conformal time. We also have the Friedmann constraint equation

$$h^2 = \frac{1}{12} \left( \phi'^2 + \beta'^2 + e^{2\lambda\phi} \sigma'^2 \right) . \quad (3.6)$$

It is straightforward to derive the general dilaton-moduli-vacuum solutions where  $\sigma' = 0$ . All the remaining equations of motion can be integrated directly and one obtains

$$\tilde{a} = \tilde{a}_* \left| \frac{\eta}{\eta_*} \right|^{1/2} , \quad (3.7)$$

$$e^\phi = e^{\phi_*} \left| \frac{\eta}{\eta_*} \right|^{r_\pm} , \quad (3.8)$$

$$e^\beta = e^{\beta_*} \left| \frac{\eta}{\eta_*} \right|^s . \quad (3.9)$$

The Friedmann constraint equation requires  $r_{\pm} = \pm\sqrt{3-s^2}$ .

These dilaton-moduli vacuum solutions are monotonic power-law solutions. The solutions with  $r_+ > 0$  approach weak coupling ( $\phi \rightarrow -\infty$ ) as  $\eta \rightarrow 0$  and strong coupling ( $\phi \rightarrow +\infty$ ) as  $\eta \rightarrow \pm\infty$ , while the reverse happens on the  $r_- < 0$  solutions. The  $r_+$  and  $r_-$  solutions are related by the S-duality  $\phi \rightarrow -\phi$ . However, for either  $r_+$  or  $r_-$ , the solutions in the Einstein frame always evolve from a low curvature regime as  $\eta \rightarrow -\infty$  towards a high curvature regime as  $\eta \rightarrow 0$ , or from a high curvature regime as  $\eta \rightarrow 0$  to a low curvature regime as  $\eta \rightarrow +\infty$ .

The general  $SL(2, \mathbb{R})$  transformation given in Eqs. (2.10) and (2.11) then allows one to write down the general solution for a spatially flat FRW universe with  $\sigma' \neq 0$  [17, 13, 18]:

$$\tilde{a} = \tilde{a}_* \left| \frac{\eta}{\eta_*} \right|^{1/2}, \quad (3.10)$$

$$e^{\phi} = \frac{e^{\phi_*}}{2^{1/\lambda}} \left[ \left| \frac{\eta}{\eta_*} \right|^{-\lambda r} + \left| \frac{\eta}{\eta_*} \right|^{\lambda r} \right]^{1/\lambda}, \quad (3.11)$$

$$\sigma = \sigma_* \pm \frac{e^{-\lambda\phi_*}}{\lambda} \left[ \frac{|\eta/\eta_*|^{-\lambda r} - |\eta/\eta_*|^{\lambda r}}{|\eta/\eta_*|^{-\lambda r} + |\eta/\eta_*|^{\lambda r}} \right], \quad (3.12)$$

$$e^{\beta} = e^{\beta_*} \left| \frac{\eta}{\eta_*} \right|^s, \quad (3.13)$$

where we still have  $r^2 + s^2 = 3$ , although the sign of  $r$  is now irrelevant and we shall choose  $r > 0$  in what follows.

The evolution of the axion and dilaton in the Einstein frame is determined solely by the constant  $\lambda$ . The asymptotic form of the solutions in the early or late times limits (either  $\eta \ll \eta_*$  or  $\eta \gg \eta_*$ ) is determined by the dilaton-moduli-vacuum solutions given in Eqs. (3.7)–(3.9), where  $\sigma' \rightarrow 0$ .

For  $\lambda > 0$  the general axion-dilaton-moduli solution approaches the dilaton-moduli-vacuum solution with  $r_- = -r$  when  $\eta \rightarrow 0$ . The behaviour for  $|\eta| \gg |\eta_*|$  is given by the (duality related) vacuum solution with  $r_+ = +r$ . The effective gravitational coupling  $G_{\text{eff}} \propto e^{\Phi} = e^{\phi/\sqrt{3+2\omega}}$  is bounded from below. It is divergent as  $\eta \rightarrow 0$  and decreases to a minimum value at a time  $\eta = \eta_*$ . It then increases indefinitely as  $|\eta| \rightarrow \infty$ . The lower bound on the coupling is determined by the canonical momentum of the axion field.

The qualitative behaviour of the cosmology is similar to that derived originally for the string model when the axion is dual to the NS-NS three-form field strength and  $\lambda = 1$  [13]. Our analysis shows that the qualitative behaviour does not depend too sensitively on the coupling between the dilaton and axion fields and is therefore quite generic in theories of this type. However the asymptotic limits are interchanged for  $\lambda < 0$ . The  $r_+ = +r$  solution applies for  $|\eta| \ll |\eta_*|$  and the  $r_- = -r$  branch at  $|\eta| \gg |\eta_*|$ . The coupling  $\phi$  is now bounded from above. The effective gravitational

coupling  $e^\Phi$  increases from zero at  $\eta \rightarrow 0$ , reaches a maximum at  $\eta = \eta_*$ , and decreases back to zero as  $|\eta| \rightarrow \infty$ .

The value of  $\sqrt{3+2\omega}$  is important in determining the behaviour of the scale factor in the Jordan frame. The conformal transformation presented in Eq. (2.1) leads to a relation between the cosmological scale factor,  $a$ , in the Jordan frame and the scale factor in the Einstein frame,

$$a = e^{\phi/(2\sqrt{3+2\omega})} \tilde{a}. \quad (3.14)$$

Using Eqs. (3.7) and (3.8) then implies that when  $\sigma' = 0$  the scale factor in the Jordan frame is

$$a = \tilde{a}_* e^{\phi_*/(2\sqrt{3+2\omega})} \left| \frac{\eta}{\eta_*} \right|^{(1+r_\pm/\sqrt{3+2\omega})/2}. \quad (3.15)$$

For  $\sigma' \neq 0$  the Jordan frame scale factor evolves as

$$a = a_* \left| \frac{\eta}{\eta_*} \right|^{1/2} \left[ \left| \frac{\eta}{\eta_*} \right|^{\lambda r} + \left| \frac{\eta}{\eta_*} \right|^{-\lambda r} \right]^{1/(2\lambda\sqrt{3+2\omega})}. \quad (3.16)$$

For  $\lambda > 0$  and  $r > \sqrt{3+2\omega}$  the scale factor in the Jordan frame is infinitely large as  $\eta \rightarrow 0$  and the Universe undergoes an accelerated contraction for  $\eta > 0$ . It reaches a minimum size before re-expanding to infinity as  $\eta \rightarrow \infty$ . If in addition  $r \geq 3\sqrt{3+2\omega}$  then the Ricci scalar remains finite at  $\eta \rightarrow 0$  and the evolution becomes non-singular in the Jordan frame<sup>3</sup>. For  $\lambda > 0$  but  $r < \sqrt{3+2\omega}$  the universe has zero spatial volume as  $\eta \rightarrow 0$  and expands out of the singularity. For  $\lambda < 0$  there are no non-singular solutions and for  $r > \sqrt{3+2\omega}$  there is an upper bound on the maximum size attained by the universe and it undergoes recollapse after a finite proper time.

Note that in all these solutions there is no cosmological inflation in the Einstein frame in the sense that  $\ddot{\tilde{a}}/\tilde{a} = h' < 0$  at all times. This must be the case since massless scalar fields do not violate the dominant energy condition. However, the solutions for  $\eta < 0$  do share many of the useful properties of inflation [1]. In particular the comoving Hubble length,  $1/h = 2\eta$ , decreases as  $\eta \rightarrow 0$ . This in principle could allow one to explain the homogeneity of the universe on scales above the Hubble length by causal physics and this is the basis of the pre-big bang scenario.

In terms of the low-energy string effective action the pre-big bang epoch [1] is simply a collapsing universe in the Einstein frame. It is anticipated that in string theory the curvature singularity which is the usual endpoint of collapse in general relativity should be avoided due to the large-scale/small-scale T-duality which implies a minimal length scale. If this can smoothly connect the collapsing pre-big bang branch to the post-big bang expanding universe, then string theory could perhaps explain the large-scale structure of our present universe.

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<sup>3</sup>Similar behaviour is found in the axion frame of the low-energy string effective action [4].

## 4 Linear perturbations

In this Section we consider linear perturbations with comoving wavenumber  $k$  about the homogeneous FRW solutions presented in the previous Section. Working in the uniform curvature gauge, we can write the linearised equations of motion as

$$\delta\phi'' + 2h\delta\phi' + k^2\delta\phi = 2\lambda e^{2\lambda\phi} [\sigma'\delta\sigma' + \lambda\sigma'^2\delta\phi] , \quad (4.1)$$

$$\delta\sigma'' + 2h\delta\sigma' + k^2\delta\sigma = -2\lambda [\phi'\delta\sigma' + \sigma'\delta\phi'] , \quad (4.2)$$

$$\delta\beta'' + 2h\delta\beta' + k^2\delta\beta = 0 . \quad (4.3)$$

The curvature perturbation is given from the constraint equation

$$\zeta = \frac{1}{12h} (\phi'\delta\phi + \beta'\delta\beta + e^{2\lambda\phi}\sigma'\delta\sigma) , \quad (4.4)$$

where  $\zeta$  gives the curvature perturbation on uniform energy density hypersurfaces as  $k\eta \rightarrow 0$  [4, 21]. Technical details regarding the choice of gauge and definition of metric perturbations were presented in [4].

Although we have a non-trivial coupling between the axion and dilaton fields, we can in fact integrate the equations of motion to yield analytic solutions for the linear perturbations, even when the background fields evolve as given by Eqs. (3.10–3.13). To do so we must exploit the symmetry of the Lagrange density under S-duality.

We construct explicitly S-duality invariant perturbations

$$u \equiv \frac{\tilde{a}}{2\lambda^2 h} \text{tr}(JN'J\delta N) , \quad (4.5)$$

$$= \frac{\tilde{a}}{h} (\phi'\delta\phi + e^{2\lambda\phi}\sigma'\delta\sigma) , \quad (4.6)$$

$$v \equiv \frac{\tilde{a}}{2\lambda^2 h} \text{tr}(-JN.JN'J\delta N) , \quad (4.7)$$

$$= \frac{e^{\lambda\phi}\tilde{a}}{h} (\phi'\delta\sigma - \sigma'\delta\phi) . \quad (4.8)$$

The equations of motion for the fields then reduce to

$$u'' + \left[ k^2 + \frac{1}{4\eta^2} \right] u = 0 , \quad (4.9)$$

$$v'' + \left[ k^2 + \frac{1 - 4\lambda^2 r^2}{4\eta^2} \right] v = 0 . \quad (4.10)$$

The general solution for the axion-dilaton perturbations is thus given by

$$u = |k\eta|^{1/2} [u_+ H_0^{(1)}(|k\eta|) + u_- H_0^{(2)}(|k\eta|)] , \quad (4.11)$$

$$v = |k\eta|^{1/2} [v_+ H_{|\lambda r|}^{(1)}(|k\eta|) + v_- H_{|\lambda r|}^{(2)}(|k\eta|)] , \quad (4.12)$$



where  $H_\nu^{(i)}$  are Hankel functions of the first or second kind of order  $\nu$ . This displays the classic behaviour expected of perturbations in a FRW model. For small scales “within the horizon” (i.e.,  $|k\eta| \gg 1$ ) the solutions oscillate, while for  $|k\eta| \ll 1$  they cease oscillating as  $|k\eta| \rightarrow 0$ .

Perturbations in the modulus field remain decoupled from the axion and dilaton fields and their equation of motion can be written as

$$w'' + \left[ k^2 + \frac{1}{4\eta^2} \right] w = 0 , \quad (4.13)$$

where

$$w \equiv \frac{\tilde{a}}{h} \beta' \delta \beta . \quad (4.14)$$

The general solution for linear perturbations in the modulus field is thus given by

$$w = |k\eta|^{1/2} \left[ w_+ H_0^{(1)}(|k\eta|) + w_- H_0^{(2)}(|k\eta|) \right] . \quad (4.15)$$

The curvature perturbation  $\zeta$  is then given by the constraint equation (4.4) as

$$\zeta = \frac{u + w}{12\tilde{a}} \quad (4.16)$$

and is manifestly S-duality invariant. Because  $v$  does not contribute to  $\zeta$  we can consider it as an isocurvature perturbation.

For solutions with  $\eta > 0$  we have a priori no means by which to determine the constants  $u_\pm$ ,  $v_\pm$  and  $w_\pm$ . All modes start outside the horizon at the singularity as  $\eta \rightarrow 0$ , so there is no causal mechanism that can establish a particular set of initial conditions. However for solutions with  $\eta < 0$ , all modes start within the horizon at early times as  $\eta \rightarrow -\infty$  and so we may reasonably assume that at sufficiently early times,  $|k\eta| \gg 1$ , they are in their vacuum state.

Note that as  $|\eta| \rightarrow \infty$  the background solutions approach the  $r_\pm = \pm r$  dilaton–moduli–vacuum solution where  $\sigma' \rightarrow 0$  and hence, using Eqs. (3.7) and (3.8),

$$u \rightarrow \pm 2r\tilde{a}\delta\phi , \quad (4.17)$$

$$v \rightarrow \pm 2re^{\lambda\phi}\tilde{a}\delta\sigma . \quad (4.18)$$

Thus  $u$  and  $v$  reduce to the dilaton and axion perturbations about the dilaton–vacuum solutions. Using the Minkowski vacuum state to normalise these in the low curvature regime as  $k\eta \rightarrow -\infty$  yields [4, 22]

$$u_+ = \pm 2r e^{i\pi/4} \frac{\sqrt{\pi}}{2\sqrt{k}} , \quad u_- = 0 , \quad (4.19)$$

$$v_+ = \pm 2r e^{i(2|r\lambda|+1)\pi/4} \frac{\sqrt{\pi}}{2\sqrt{k}} , \quad v_- = 0 . \quad (4.20)$$

At late times as  $\eta \rightarrow 0$  we again find  $\sigma' \rightarrow 0$  and the background solutions approach the vacuum solutions with  $r_{\mp} = \mp r$ . Thus  $u$  and  $v$  again reduce to the dilaton and axion perturbations and at late times for the  $\eta < 0$  solutions we can write down the power spectra for  $-k\eta \ll 1$ :

$$\mathcal{P}_{\delta\phi} \rightarrow \frac{2}{\pi^3} \tilde{H}^2 (-k\eta)^3 [\ln(-k\eta)]^2, \quad (4.21)$$

$$\mathcal{P}_{\delta\sigma} \rightarrow \left( \frac{C(|\lambda r|)}{2\pi} \right)^2 \left( \frac{k}{e^{\lambda\phi} \tilde{a}} \right)^2 (-k\eta)^{1-2|\lambda r|}, \quad (4.22)$$

where  $\tilde{H} \equiv h/\tilde{a}$  is the Hubble expansion rate in the Einstein frame. The numerical coefficient

$$C(x) \equiv \frac{2^x \Gamma(x)}{2^{3/2} \Gamma(3/2)} \quad (4.23)$$

approaches unity as  $x \rightarrow 3/2$ . Similarly we find

$$\mathcal{P}_{\delta\beta} \rightarrow \frac{2}{\pi^3} \tilde{H}^2 (-k\eta)^3 [\ln(-k\eta)]^2 \quad (4.24)$$

for the modulus field.

In the pre-big bang scenario the linear perturbations are normalised in the low curvature, weakly-coupled regime as  $\eta \rightarrow -\infty$ . The pre-big bang era then establishes perturbations on all scales outside the horizon as  $\eta \rightarrow 0$ . Assuming there is a rapid, smooth transition to the post-big bang epoch this perturbation spectrum then determines the large-scale structure of the  $\eta > 0$  universe.

The spectral index of the perturbations is conventionally denoted by  $n_x$  where

$$n_x - 1 \equiv \frac{d\mathcal{P}_{\delta x}}{d \ln k}, \quad (4.25)$$

and a scale-invariant Harrison-Zel'dovich spectrum corresponds to  $n_x = 1$ . Thus for the dilaton and axion perturbations as  $\eta \rightarrow 0$  we find from Eqs. (4.21) and (4.22) that

$$n_{\phi} = 4, \quad (4.26)$$

$$n_{\sigma} = 4 - 2|\lambda r| = 4 - 2 \left| \lambda \sqrt{3 - s^2} \right|, \quad (4.27)$$

$$n_{\beta} = 4. \quad (4.28)$$

Note that unlike the dilaton and modulus spectrum, the axion spectrum can be consistent with scale invariance. We employ these results in the following section to derive the perturbation spectra generated in a particular string inspired model.

## 5 A string inspired example

Ten-dimensional supergravity theories represent the low energy limit of string theories. The NS-NS bosonic sector for theories of this type is

$$S = \int d^{10}x \sqrt{-g} e^{-\hat{\Phi}} \left[ R + (\nabla \hat{\Phi})^2 - \frac{1}{12} H^2 \right] , \quad (5.1)$$

where  $H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]}$  is the rank-three antisymmetric tensor field strength and  $\hat{\Phi}$  is the dilaton field. This can be dimensionally reduced to four dimensions by compactifying on an isotropic six-dimensional torus [15, 16], where the ten-dimensional metric is given by

$$g_{MN} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & \exp[\beta(x)/\sqrt{3}] \delta_{mn} \end{pmatrix} , \quad (5.2)$$

$x$  represents the four-dimensional coordinates, and  $\mu, \nu, \dots (m, n, \dots)$  are the four- (six-) dimensional indices. It is assumed that the components of the antisymmetric tensor field are given by  $B_{\mu\nu} = B_{\mu\nu}(x)$  and  $B_{mn} = \alpha(x) \epsilon_{mn}$  [15]. It is further assumed that  $\hat{\Phi} = \hat{\Phi}(x)$ .

The effective four-dimensional action can then be written as

$$S = \int d^4x \sqrt{-g} e^{-\phi} \left[ R + (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla \sigma)^2 - \frac{1}{2} (\nabla \beta)^2 - \frac{1}{2} e^{-2\beta/\sqrt{3}} (\nabla \alpha)^2 \right] , \quad (5.3)$$

where  $\phi \equiv \hat{\Phi} - \sqrt{3}\beta$  is the shifted dilaton field and the antisymmetric tensor field has been expressed in terms of its dual, the pseudo-scalar axion field  $\sigma$ , i.e.,  $H^{\mu\nu\lambda} = e^{\phi} \epsilon^{\mu\nu\lambda\rho} \nabla_{\rho} \sigma$ , where  $\epsilon^{\mu\nu\lambda\rho}$  is the antisymmetric four-form satisfying  $\nabla_{\kappa} \epsilon^{\mu\nu\lambda\rho} = 0$ .

By performing the conformal transformation Eq. (2.1) to the Einstein frame, it can be seen that the fields  $\phi$  and  $\sigma$  parameterise an  $\text{SL}(2, \mathbb{R})/\text{U}(1)$  coset as in the action Eq. (2.3) with  $\lambda = 1$ :

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - \frac{1}{2} (\tilde{\nabla} \phi)^2 - \frac{1}{2} e^{2\phi} (\tilde{\nabla} \sigma)^2 - \frac{1}{2} (\tilde{\nabla} \beta)^2 - \frac{1}{2} e^{-2\beta/\sqrt{3}} (\tilde{\nabla} \alpha)^2 \right] . \quad (5.4)$$

However, the reflection symmetry of the modulus field  $\beta$  is extended. The moduli fields  $\alpha$  and  $\beta$  together parameterise another  $\text{SL}(2, \mathbb{R})/\text{U}(1)$  coset with an effective coupling  $\lambda = -1/\sqrt{3}$ . The action Eq. (5.4) may therefore be written as

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} + \frac{1}{4} \text{Tr} (\tilde{\nabla} N \tilde{\nabla} N^{-1}) + \frac{3}{4} \text{Tr} (\tilde{\nabla} P \tilde{\nabla} P^{-1}) \right] , \quad (5.5)$$

where  $N$  is given by Eq. (2.5) with  $\lambda = 1$ , and

$$P = \begin{pmatrix} e^{-\beta/\sqrt{3}} & \frac{1}{\sqrt{3}} \alpha e^{-\beta/\sqrt{3}} \\ \frac{1}{\sqrt{3}} \alpha e^{-\beta/\sqrt{3}} & e^{\beta/\sqrt{3}} + \frac{1}{3} \alpha^2 e^{-\beta/\sqrt{3}} \end{pmatrix} . \quad (5.6)$$

The Lagrange density in the Einstein frame is therefore invariant under two independent  $SL(2, R)$  transformations

$$\tilde{g}_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} , \quad N \rightarrow \Theta N \Theta^T , \quad P \rightarrow \theta P \theta^T , \quad (5.7)$$

where  $\Theta$  and  $\theta$  are two independent  $SL(2, R)$  matrices.

When  $\alpha$  is frozen, the action Eq. (5.5) reduces to Eq. (2.3) with  $\lambda = 1$ . Thus the general cosmological solution for  $\alpha' = 0$  is given by Eqs. (3.10)–(3.13). The  $SL(2, R)$  transformation  $P \rightarrow \theta P \theta^T$  then yields the general moduli field solutions with  $\alpha' \neq 0$ ,

$$e^\beta = 2^{\sqrt{3}} e^{\beta_*} \left[ \left| \frac{\eta}{\eta_*} \right|^{-s/\sqrt{3}} + \left| \frac{\eta}{\eta_*} \right|^{s/\sqrt{3}} \right]^{-\sqrt{3}} , \quad (5.8)$$

$$\alpha = \alpha_* \pm \sqrt{3} e^{\beta_*/\sqrt{3}} \left[ \frac{|\eta/\eta_*|^{-s/\sqrt{3}} - |\eta/\eta_*|^{s/\sqrt{3}}}{|\eta/\eta_*|^{-s/\sqrt{3}} + |\eta/\eta_*|^{s/\sqrt{3}}} \right] . \quad (5.9)$$

The transformation mixes these two fields in a non-trivial fashion and results in a time-dependent  $\alpha$ , but it leaves the other fields, and hence the metric in the string frame as well as the Einstein frame, invariant. This follows since the combined role of  $\{\alpha, \beta\}$  is effectively that of the single modulus field  $\beta$  present in action Eq. (1.1).

To study linear perturbations about these homogeneous solutions we should pick quantities invariant under the duality transformations in Eq. (5.7). Thus we have  $u$  and  $v$  as defined in Eqs. (4.5) and (4.7), with  $\lambda = 1$ , and

$$w \equiv \frac{3\tilde{a}}{2h} \text{tr}(JP'J\delta P) , \quad (5.10)$$

$$= \frac{\tilde{a}}{h} (\beta' \delta \beta + e^{-2\beta/\sqrt{3}} \alpha' \delta \alpha) , \quad (5.11)$$

$$x \equiv \frac{3\tilde{a}}{2h} \text{tr}(-JPJP'J\delta P) , \quad (5.12)$$

$$= \frac{e^{-\beta/\sqrt{3}} \tilde{a}}{h} (\beta' \delta \alpha - \alpha' \delta \beta) . \quad (5.13)$$

The curvature perturbation  $\zeta$  is still given by Eq. (4.16), but  $w$  is now given by Eq. (5.10).

By analogy with the axion and dilaton perturbations in Section 4, we can immediately write down the spectral indices of the perturbations in this model. We find

$$n_\phi = 4 , \quad n_\sigma = 4 - 2r , \quad (5.14)$$

$$n_\beta = 4 , \quad n_\alpha = 4 - \frac{2}{\sqrt{3}} s , \quad (5.15)$$

where  $r^2 + s^2 = 3$ . It is worth noting that the constraint between  $r$  and  $s$  has the effect of linking the indices for the two axions  $\sigma$  and  $\alpha$ . As  $r$  ranges from 0 to  $\sqrt{3}$ , we find  $n_\sigma$  ranges from 4 to 0.5 and  $n_\alpha$  from 2 to 4.

## 6 Discussion

In this paper we have investigated the evolution of dilaton, moduli and axion (curvature and isocurvature) perturbations in a general class of models with an arbitrary coupling,  $\lambda$ , between the dilaton field and the axion field. Such models possess an  $SL(2,R)$  symmetry between the dilaton and axion fields making it possible to solve the full homogeneous equations of motion for the system. Armed with these solutions we solved the linear perturbation equations for the fields, making use of the fact that we could write down unique S-duality invariant combinations of perturbed fields. In a pre-big bang type scenario the dilaton and modulus spectra, and thus from Eq. (4.16) the curvature perturbation, is strongly tilted towards small scales, leaving an almost perfectly homogeneous universe on large scales. While this is consistent with the assumption of a homogeneous background, it does not provide a spectrum of perturbations on large scales capable of seeding the density fluctuations observed today. However the perturbation spectrum of the axion field is scale invariant when  $|\lambda|r = 3/2$ , where  $r$  is an integration constant satisfying  $0 \leq r \leq \sqrt{3}$ . The bounds on the possible values of the spectral index are therefore

$$4 - 2\sqrt{3}|\lambda| \leq n_\sigma \leq 4. \quad (6.1)$$

Hence it is possible to obtain  $n_\sigma \leq 1$  for  $|\lambda| \geq \sqrt{3}/2$ .

String theory offers many possibilities in terms of the low-energy four-dimensional theory. Axion type fields can arise from the compactification of the antisymmetric three-form field strength on the internal space. We have investigated the consequences of an isotropic compactification where the extra axion,  $\alpha$ , couples to the modulus field such that the full action has an  $SL(2,R) \times SL(2,R)$  symmetry. This enables us to calculate the evolution of the perturbations in the more complicated system. Whereas the dilaton and moduli spectra produced in the pre-big bang scenario are too steep, both axion fields could have less steep spectra. Their spectral indices are linked through the Friedmann constraint equation. In particular as  $r$  ranges from 0 to  $\sqrt{3}$ ,  $n_\sigma$  ranges from 4 to 0.5 and  $n_\alpha$  from 2 to 4. Such a result indicates that only the  $\sigma$  field can possess a scale-invariant spectrum in this model.

The perturbation spectra are generated in the pre-big bang phase. Perturbations inherited by the post-big bang phase will depend on the mechanism by which the transition between the two phases proceeds. However, despite this caveat, it is important that in principle it is possible to generate observationally interesting power spectra in this scenario, and it demonstrates the importance of axion fields in the pre-big bang picture.

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